The production rate of the coarse grained Gibbs entropy and the Kolmogorov-Sinai entropy: a real connection ?

Massimo Falcioni and Luigi Palatella
Dipartimento di Fisica and Istituto dei Sistemi Complessi del Consiglio Nazionale delle Ricerche, Università di Roma "La Sapienza", P.le A.Moro 2, Rome 00185, Italy

Angelo Vulpiani
Dipartimento di Fisica and Istituto dei Sistemi Complessi del Consiglio Nazionale delle Ricerche, Università di Roma "La Sapienza", P.le A.Moro 2, Rome 00185, Italy and INFN, Sezione di Roma "La Sapienza"

We discuss the connection between the Kolmogorov-Sinai entropy, h_{KS}, and the production rate of the coarse grained Gibbs entropy, r_G. Detailed numerical computations show that the (often accepted) identification of the two quantities does not hold in systems with intermittent behavior and/or very different characteristic times and in systems presenting pseudo-chaos. The basic reason of this fact is in the asymptotic (with respect to time) nature of h_{KS}, while r_G is a quantity related to short time features of a system.

PACS numbers: 05.45.-a, 45.05.+x, 05.70.Ln

I. INTRODUCTION

The physical entropy is a quantity that plays a key role in the understanding of the basic laws ruling the macroscopic behavior of systems with many degrees of freedom. We just mention the Boltzmann's microscopic interpretation of the macroscopic Clausius equilibrium entropy, and the celebrated H-theorem [1].

On the other hand the term entropy is widely used also in contexts different from thermodynamics and statistical mechanics. In the information theory there is the Shannon entropy [2, 3, 4], while in dynamical systems one uses the Kolmogorov-Sinai entropy (and other entropic quantities like, *e.g.*, the Renyi entropies) [4, 5].

The connection between some properties of the non-equilibrium thermodynamical systems and the underlying chaotic dynamics, recently, has attracted the interest of many scientists [6, 7, 8, 9]. The main question is if and how the dynamical characteristic quantities (such as Lyapunov exponents or Kolmogorov-Sinai entropy) are related to macroscopic physical properties (*e.g.*: diffusion coefficients and entropy production rate) [6, 7, 8, 9, 10, 11, 12, 13, 14, 15].

Some authors, on the basis of reasonable arguments and numerical computations on simple dynamical systems, claim the existence of a relation between the Kolmogorov-Sinai entropy, h_{KS}, and the production rate, r_G, of a suitably averaged coarse-grained Gibbs entropy [16].

The main goal of this work is to show that this is not the generic case. To this end we will study two different kinds of discrete time systems:

a) one-dimensional intermittent maps, with local Lyapunov exponents very different from the mean value;

b) slightly coupled maps with very different (uncoupled) Lyapunov exponents.

We will show that when there are different (local Lyapunov exponents) time scales in the different regions of the phase space, there may be no room for an identification, on a meaningful time interval, between h_{KS} and r_G. This is so because, when averaging *at a fixed time* the entropy contributions originating from regions with different time-scales, densities with different space-scales get mixed, at possibly different relaxation stages.

A second point we want to stress is that, even if a linearly increasing time behavior of the coarse-grained entropy is observed, the rate of growth is not necessarily given by the Kolmogorov-Sinai invariant of the system. This will be shown by studying discretized versions of a chaotic map (i.e. an automaton). In such a case, the memory of the chaotic character of the original system may allow for a *transient* chaotic-like regime long enough to make the behavior of the strictly periodic system indistinguishable from the one of its truly chaotic ancestor. That is, pseudochaos is at work [17, 18]: the long time properties of the system, like $h_{KS}(= 0)$, remain hidden and do not affect quantities, like $r_G(\neq 0)$, related to short time features of the system.

The paper is organized as follows. In Sect. 2 we recall some basic concepts and methods in chaotic dynamical systems and statistical mechanics. In addition we give a simple argument for the connection between h_{KS} and r_G, stressing the weak points of the argument. In Sect. 3 we discuss the results of numerical computation for the time evolution of the coarse-grained Gibbs entropy in systems with "non trivial" dynamical features, *i.e.* with intermittency and different characteristic times. Sect. 4 is devoted to conclusions and discussions.

II. A BRIEF OVERVIEW OF BASIC FACTS

Since the pioneering work of Kolmogorov [3, 5], the relevance of the Kolmogorov-Sinai entropy for a proper characterization of the behavior of a chaotic dynamical system was clear. To perform the computation of h_{KS} one has to choose a partition $\mathcal{A}$ of the phase space and to assign each cell of the partition an integer value i. In such a way, the trajectories of a dynamical system, with continuous states, sampled at discrete times,

$$\mathbf{x}(1), \mathbf{x}(2), ..., \mathbf{x}(j), ..., \mathbf{x}(T)$$

become symbolic sequences

$$i(1), i(2), ..., i(j), ..., i(T) \tag{1}$$

whose meaning is that at time j the trajectory is in the cell labeled by $i(j)$. Then one defines the probability of each word (or block) of length n, $p^{(\mathcal{A})}(k_1, k_2, .., k_n)$, counting how many times one meets the word $k_1, k_2, ..., k_n$ along the sequence (1). The entropy of the blocks of size n, $H^{(\mathcal{A})}(n)$, thus reads

$$H^{(\mathcal{A})}(n) = -\sum_{k_1,..,k_n} p^{(\mathcal{A})}(k_1, .., k_n) \log p^{(\mathcal{A})}(k_1, .., k_n). \tag{2}$$

The Kolmogorov-Sinai entropy is defined by the sup over all partitions of the asymptotic value of the rate of increase of $H^{(\mathcal{A})}(n)$, i.e.

$$h_{KS} = \sup_{\mathcal{A}} \lim_{n\to\infty} \frac{H^{(\mathcal{A})}(n)}{n} \,. \tag{3}$$

The quantity h_{KS} is a numerical invariant that gives a good characterization of a chaotic system, but unfortunately it is almost impossible to compute analytically (except very few simple cases) and also rather difficult from a numerical point a view. From a physical point of view it is rather natural to use regular partitions with hyper-cubic cells of edge ϵ. Let us denote with $H^{(\epsilon)}(n)$ the n-block entropy on a partition of this kind and ϵ-entropy the limit

$$h(\epsilon) = \lim_{n\to\infty} \frac{H^{(\epsilon)}(n)}{n} = \lim_{n\to\infty} H^{(\epsilon)}(n+1) - H^{(\epsilon)}(n) \,. \tag{4}$$

It is a remarkable fact that $h(\epsilon)$, computed with different values of ϵ, can give very interesting information about the properties of the system[4, 5]. Moreover it is possible to obtain h_{KS} by considering the limit $\epsilon \to 0$ in (4), instead of the sup operation in (3). By recalling the Shannon-McMillan equipartition theorem (stating that, for large n, the number of "typical" n-words increases as $\exp[h(\epsilon)\, n]$) one has that $h(\epsilon)$ gives the (asymptotic in time) exponentially growing rate of the number of typical trajectories of the system, in the limit of high resolution (as measured by ϵ). It is intuitive that this growing rate must be linked to the exponentially fast separation of nearby trajectories. The Pesin theorem is the rigorous statement of this idea: all the expanding directions contribute to the diversification of the trajectories and to the increase of their number

$$h_{KS} = \sum_{\lambda_i > 0} \lambda_i \,, \tag{5}$$

where $(\lambda_i > 0)$ means the sum over positive λ_i. Pesin's identity (5) provides us with a useful alternative way to compute h_{KS}. The Lyapunov exponents can be numerically computed without particular difficulties, even in high-dimensional systems; on the contrary, because of the exponential proliferation of the n-words, the Kolmogorov-Sinai entropy becomes rapidly unattainable by numerical methods (a part low-dimensional systems). Therefore very often the Pesin formula is basically the unique way to compute h_{KS}.

Notice that h_{KS} is an entropy rate defined on the ensemble of the trajectories of a system (according to some stationary probability measure), even if, by means of ergodicity, it is practically computed using only one single long trajectory. Qualitative arguments may be given [19] to support a connection between h_{KS} and the rate of variation of an entropy-like quantity, that we call Gibbs entropy, defined on the phase space of the system. A possible line of reasoning is the following.

Consider a deterministic dynamical law

$$\mathbf{x} \to T^t \mathbf{x} \tag{6}$$

(where $\mathbf{x}$ is a D-dimensional vector) and a probability density $\rho(\mathbf{x}, t)$, that gives a distribution of states of the system throughout its phase space at a time t. We define the Gibbs entropy of ρ as follows

$$S(\rho_t) = -\int \rho(\mathbf{x}, t) \ln[\rho(\mathbf{x}, t)] d\mathbf{x} \,, \tag{7}$$

i.e. its conditional entropy with respect to a uniform density. For chaotic dissipative systems, where $\rho(\mathbf{x}, t)$ tends to a singular (fractal) measure, definition (7) becomes meaningless. Nevertheless, following a nice idea of Ruelle [20], one can avoid this difficulty, simply by adding (or considering unavoidably present) a small noise term in the evolution law: in such a way one obtains a $\rho(\mathbf{x}, t)$ continuous with respect to the Lebesgue measure. If $J(\mathbf{x}, t)$ is the Jacobian of (6), a straightforward computation gives

$$S(\rho_t) = S(\rho_0) + \int \rho(\mathbf{x}, t) \ln |J(\mathbf{x}, t)| d\mathbf{x} \,. \tag{8}$$

In the case of volume conserving evolutions, one has: $S(\rho_t) = S(\rho_0)$. To allow for an entropy variation one needs a coarse-graining. Let us consider a hyper-cubic partition, as introduced above, and let us define the probability $P^\epsilon(i, t)$ to find the state of the system in the cell i at time t:

$$P^\epsilon(i, t) = \int_{\Lambda_i^\epsilon} \rho(\mathbf{x}, t) d\mathbf{x} \tag{9}$$

where Λ_i^ϵ is the region singled out by the $i-$th cell. Let us introduce the ϵ-coarse-grained Gibbs entropy

$$S^\epsilon(P_t) = -\sum_i P^\epsilon(i,t) \ln P^\epsilon(i,t).$$

If ϵ is small enough $S(\rho_t)$ and $S^\epsilon(P_t)$ are trivially related:

$$S^\epsilon(P_t) \simeq S(\rho_t) + D \ln\left(\frac{1}{\epsilon}\right).$$

If one considers a distribution of initial conditions that is different from zero only over one (or very few) cell(s), one has, for ϵ small enough and a time not too short,

$$S^\epsilon(P_t) = S^\epsilon(P_0) + h_{KS} t\,. \tag{10}$$

One can argue as follows. Assume that the system has m positive Lyapunov exponents and that $\rho(\mathbf{x},0)$ is localized around $\mathbf{x}^c(0)$. In a suitable reference system (with the axes along the eigendirections of the Lyapunov exponents), if $\rho(\mathbf{x},0)$ has a Gaussian shape, $\rho(\mathbf{x},t)$, for some times, is still well approximated by a Gaussian with variances

$$\sigma_j^2(t) = \sigma_j^2(0) \exp\{2\lambda_j t\} \tag{11}$$

therefore:

$$\rho(\mathbf{x},t) \simeq \prod_{j=1}^{D} \frac{1}{\sqrt{2\pi\sigma_j^2(t)}} e^{-\frac{(x_j - x_j^c(t))^2}{2\sigma_j^2(t)}} \tag{12}$$

where $\mathbf{x}^c(t)$ is the state evolved from $\mathbf{x}^c(0)$. From this, in the non-grained case, one gets

$$S(\rho_t) = S(\rho_0) + \sum_j \ln \frac{\sigma_j(t)}{\sigma_j(0)} = S(\rho_0) + \sum_{j=1}^{D} \lambda_j t\ .$$

It is clear that $S(\rho_t) = S(\rho_0)$ if the phase space volume is conserved. Considering now the coarse-graining (9), one has that along the directions of the negative Lyapunov exponents $(m+1, m+2, \dots)$, for a long enough t:

$$\sigma_k(t) \sim \sigma_k(0) e^{-|\lambda_k| t} \leq \epsilon$$

This implies that

$$P_i^\epsilon(t) \simeq \prod_{j=1}^{m} \frac{1}{\sqrt{2\pi\sigma_j^2(t)}} e^{-\frac{(x_j^{(i)} - x_j^c(t))^2}{2\sigma_j^2(t)}}$$

and therefore

$$S^\epsilon(P_t) = S^\epsilon(P_0) + \sum_{j=1}^{m} \lambda_j t\ .$$

With the aid of the Pesin's formula (5) eq. (10) follows. Let us stress that the transition from (8) to (10) is allowed by the fact that, in the presence of a coarse-graining, the contracting eigendirections (corresponding to the negative values of the Lyapunov exponents) cannot balance the effects of the expanding ones.

At this point we have to notice that, by definition, the Gibbs entropy (7) explicitly depends on the particular chosen initial density. In the discussion here above this dependence may be labeled by the cell, $\mathbf{x}^c$, where the distribution is initially different from zero. On the contrary, h_{KS} is an asymptotic global property of the system. So, one may expect that a density independent behavior, as in (10), can be found only in "friendly" dynamical systems, *i.e.* systems with no fluctuations. A generic system possesses a certain degree of intermittency, so that, for instance, the expanding and contracting properties may strongly depend on the phase space region the trajectory is visiting. This calls for an averaging over the initial condition $\mathbf{x}^c$, weighted, say, by the natural invariant measure of the system:

$$S(\rho_t) \to S(t) = \int S(t|\mathbf{x}^c) \rho_{eq}(\mathbf{x}^c) d\mathbf{x}^c \tag{13}$$

where $S(t|\mathbf{x_c})$ is $S(\rho_t)$ with $\rho_0(\mathbf{x})$ localized around $\mathbf{x_c}$. The same averaging procedure leads to $S^\epsilon(t)$ from $S^\epsilon(P_t)$. This operation yields intrinsic quantities, that can depend on global properties of the system. The interesting question is whether the simple relation (10) survives, as an observable property, for the averaged coarse-grained entropy. Notice that equation (10), besides its conceptual interest, would result in a numerically simple way to determine h_{KS}. In some systems suitable ranges of t and ϵ exist where the relation is verified [16]. However, we believe, in agreement with Ref. [15], that what has been found in Ref. [16] is just a "lucky" coincidence. With regard to that, it is important to stress that in the above presented arguments, for the derivation of (10), there are (at least) two delicate points:

a) both Lyapunov exponents and Kolmogorov-Sinai entropy are quantities defined in the limits of high resolution ($\epsilon \to 0$) and long times ($t \to \infty$);

b) a behavior like (11) holds only for "short" time, *i.e.*:

$$t \lesssim \frac{1}{\lambda_1} \ln \frac{1}{\sigma(0)}\ ;$$

and this is so also for the linear behavior in (10) that is expected to be valid for

$$t \lesssim \frac{1}{\lambda_1} \ln \frac{1}{\epsilon}.$$

If intermittency is present one has to replace λ_1 with the largest local Lyapunov exponent.

Since for a coarse-grained Gibbs entropy a finite ϵ is mandatory, it is not obvious that the previous time regimes a) and b) have a non-empty overlap. One may note that in the entropic analysis of the n-words one of

the two asymptotic limits can be relaxed: i.e. one can work with non-infinitesimal ϵ and therefore obtain the ϵ-entropy $h(\epsilon)$ which is an asymptotic in time quantity associated with a finite tolerance ϵ. However, once the size of the cells for the coarse-grained Gibbs entropy is fixed, nonetheless $\rho(x,t)$ evolves developing structures on scales $l(t) \approx \epsilon \exp(\lambda t)$ increasing in time. Therefore it is not trivial at all that a simple relation between $dS^\epsilon(t)/dt$ and $h(\epsilon)$ exists.

III. A NUMERICAL STUDY OF SIMPLE DYNAMICAL SYSTEMS.

In this section we present an entropic analysis of simple dynamical systems which show, in spite of their low dimensionality, non trivial features.

From a practical point of view the computation of $S^\epsilon(t)$ is performed as follows:

1. we select several starting conditions $\mathbf{x}_1^j(0)$, $\mathbf{x}_2^j(0)$, ..., $\mathbf{x}_N^j(0)$ all located in the j-th box of size ϵ^D
2. we let evolve all the $N \gg 1$ starting conditions up to a time t obtaining $\mathbf{x}_1^j(t), \mathbf{x}_2^j(t), ..., \mathbf{x}_N^j(t)$
3. we calculate

$$p^{\epsilon,j}(k,t) = \frac{1}{N}\sum_{i=1}^{N} \delta(\mathbf{x}_i^j(t), k)$$

with

$$\delta(\mathbf{x},k) = \begin{cases} 1 & \text{if } \mathbf{x} \in \Lambda_k^\epsilon \\ 0 & \text{otherwise} \end{cases}$$

4. we compute the entropy of $p^{\epsilon,j}(k,t)$ defined as

$$S^\epsilon(j,t) = -\sum_{k=1}^{N} p^{\epsilon,j}(k,t) \log p^{\epsilon,j}(k,t) \qquad (14)$$

5. we average this quantity on the coarse-grained invariant distribution $p_{eq}(j)$ obtained from $\rho_{eq}(\mathbf{x})$ with the procedure of Eq.(9). Thus we have

$$S^\epsilon(t) = \sum_{j=1}^{N} p_{eq}(j) S^\epsilon(j,t). \qquad (15)$$

A. Intermittent map

A recent paper [15] shows that the value of r_G in the Manneville map [21] significantly differs from the value of h_{KS}. These authors perform the analytical calculation of $S(t)$ starting from only one particular condition, namely the box containing the point $x = 0$, that in the Manneville map is the point with the lowest local Lyapunov exponent. Since in this map the value of the local Lyapunov exponent ranges from very low values ($\lambda(x) \to 0$ for $x \to 0$) to values considerably grater than 1, the authors correctly argue that the discrepancy is due to the variability of $\lambda(x)$. Nevertheless the authors of [15] do not perform the final average over the initial condition. In the following we see that the averaging procedure (15) is not able to recover the condition of Eq. (10) neither in the Manneville map nor in a less-intermittent map. Let us start with a simple case, i.e. the modified tent map given by

$$x_{t+1} = \begin{cases} x_t/p & \text{if } x_t < p \\ (1-x_t)/(1-p) & \text{if } p < x_t < 1. \end{cases} \qquad (16)$$

This map is the standard tent map if $p = 1/2$, while for small values of p one has an intermittent behavior characterized by two very different local Lyapunov exponents, namely:

$$\lambda_+ = -\log p, \;\; \lambda_- = -\log(1-p). \qquad (17)$$

The stationary distribution is constant between 0 and 1 and consequently we have

$$h_{KS} = p\lambda_+ + (1-p)\lambda_- = -p\log p - (1-p)\log(1-p) \qquad (18)$$

assuming its maximum value for $p = 1/2$ and decreasing its value for $p \to 0$. Note that this map presents an intermittent behavior, but not in a critical way like in the Manneville's map, and the distribution of the length of the "laminar" zones (the permanence in the zone $[p, 1]$) is simply exponential without power law tail. Nevertheless we see that already in this system Eq.(10) does not hold.

The numerical results are shown in fig.1. We set $\epsilon = 10^{-3}$ and we scale the time axes with the inverse of $h_{KS}(p)$, given by Eq.(18), so if Eq.(10) holds we should observe that all the curves, with different p, collapse on the straight line $S^\epsilon(t) = h_{KS}(p)t$. As one can see the agreement is good only for $p = 1/2$ (with no intermittency) while becomes worse and worse decreasing p. The main point is that the linear behavior of $S^\epsilon(t)$ should hold, according to the heuristic arguments of Ref. [16], till to a time given by

$$t_{lin} \simeq \frac{1}{\lambda_1} \log\left(\frac{1}{\epsilon}\right) \qquad (19)$$

corresponding, on the scaled time of Fig.1, to the value $-\log\epsilon \simeq 7$. This is observed for $p = 1/2$ but for small values of p one has that $S^\epsilon(t)$ increases in time with a "wrong" slope (different from h_{KS}) and later it exhibits a rather long crossover.

The origin of this effect is in the intermittent behavior of the system. Indeed the realizations $S^\epsilon(j,t)$ starting in the zone $[0, p]$ are spread on the whole interval $[0, 1]$ after few steps almost reaching the asymptotic value of $-\log\epsilon$ while the "realizations" starting for example near the unstable equilibrium point $x = 1/(2-p)$ takes several time

steps to reach the saturation giving a dominant contribution to the rate of increase of $S^\epsilon(t)$. In this way the $S^\epsilon(t)$ computed with Eq.(15) does not increase in time following the naive argument yielding to Eq.(10). The reason of the discontinuity in $S^\epsilon(1)$ for $p \to 1/2$ is explained in Appendix A.

Fig.2 shows other interesting properties of the behavior of $S^\epsilon(t)$ for $p = 0.1$ and $p = 0.4$ at varying the value of ϵ. As one can see, in the slightly intermittent case $p = 0.4$ the rescaled curves collapse together confirming the assumption of Eq.(10); while in the intermittent case $p = 0.1$ the curves do not collapse and only for very low values of ϵ a linear growth of $S^\epsilon(t)$ is present. Let us stress the fact that, at variance with the case shown in Fig. 2b, in the intermittent case (Fig.2a) the crossover regime (after the linear one and before the saturation) is very long and is comparable with the duration of the linear regime.

To connect our results to the ones of Ref. [15] we also perform the same calculation for the Manneville map given by

$$x_{t+1} = \begin{cases} x_t + k x_t^z & \text{if } x_t < d \\ (1 - x_t)/(1 - d) & \text{if } d < x_t < 1. \end{cases} \tag{20}$$

where d fulfills

$$d + k d^z = 1.$$

In the range $3/2 < z < 2$ and $k \ll 1$ there is an invariant distribution [4, 15]. Under these conditions the distribution of the permanence time τ has a power law tail with $\langle \tau \rangle < \infty$ while $\langle \tau^2 \rangle$ diverges. The only difference from the tent map calculation is that the invariant distribution $\rho_{eq}(x)$ is not constant and therefore we have to compute it numerically. Also in this case we observe a "wrong" slope (lower than h_{KS}) and an extremely long crossover behavior.

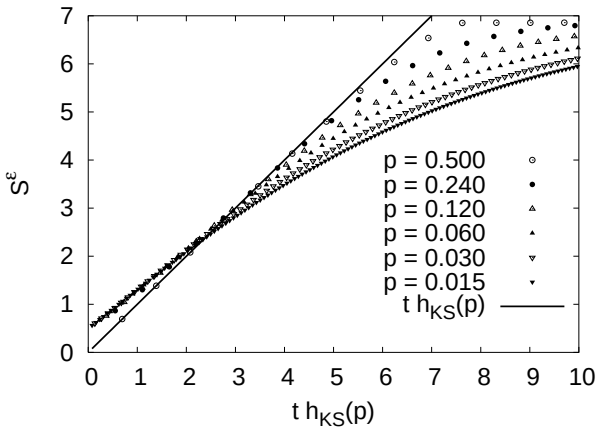

FIG. 1: $S^\epsilon(t)$ as a function of $th_{KS}(p)$ with $\epsilon = 10^{-3}$ and different values of p. Note that Eq.(10) should give the straight line $th_{KS}(p)$ for all values of p.

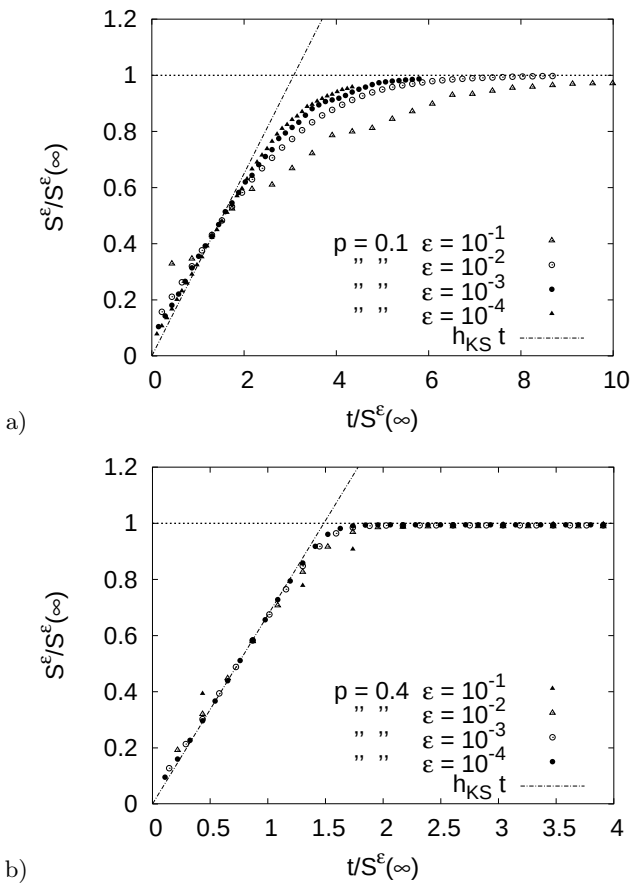

FIG. 2: $S^\epsilon(t)/S^\epsilon(\infty)$ as a function of $t/S^\epsilon(\infty)$ with different values of ϵ. In fig. a) $p = 0.1$, in fig. b) $p = 0.4$

B. 2D map

Let us now discuss a system with two different timescales. We choose the following 2D maps for the variable x_t and y_t

$$\begin{cases} x_{t+1} = & f(x_t) + \sigma \cos(2\pi(x_t + y_t)) \\ y_{t+1} = & r y_t + \sigma \cos(2\pi(x_t + y_t)) \mod 1 \end{cases} \tag{21}$$

with

$$f(x) = \begin{cases} x/p & \text{if } x < p \\ (1 - x)/(1 - p) & \text{if } p < x < 1 \end{cases} \tag{22}$$

and r is an integer greater than 1. The map for x_t is a modified tent map, like in the previous section, while the map for y_t is a generalized Bernoulli shift. We choose $p = 0.3$ and $r = 5$ in order to have two largely different uncoupled Lyapunov exponents, namely $\lambda_x \simeq 0.61$ and $\lambda_y = \log 5 \simeq 1.61$. We use this kind of coupling to avoid discontinuity at $x = 0$ or $x = 1$.

We will see that in this system the two Lyapunov exponents λ_x and λ_y have a role rather similar to that of λ_+ and λ_- for the system (16).

The coarse-grained Gibbs entropy of this 2D map is calculated starting from a 2-dimensional cell of linear

size ϵ and performing the average over the two dimensional coarse-grained invariant distribution, that also in this case turns out to be flat in the slightly coupled case. We also numerically calculated the Lyapunov exponents in the coupled case: for the small values of σ used in the numerical calculations, we observe no relevant changes from the uncoupled case. We set $\epsilon = 5 \cdot 10^{-3}$ and we study the behavior of $S^\epsilon(t)$ with different values of σ, in such a way we can study the crossover behavior at varying σ. The results for are shown in fig.3.

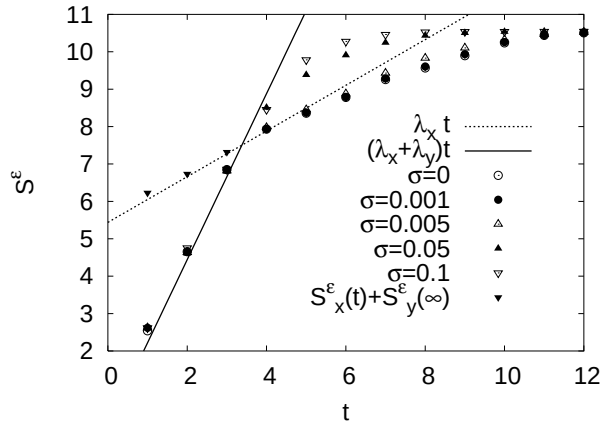

FIG. 3: $S^\epsilon(t)$ as a function of t with different values of σ, $\epsilon = 5 \cdot 10^{-3}$, $p = 0.3$, $r = 5$

As one can see, at very small t, $S^\epsilon(t)$ increases with a slope equal to the Kolmogorov-Sinai entropy $h_{KS} \simeq \lambda_x + \lambda_y = -p\log p - (1-p)\log(1-p) + \log r$. After that the phase space of the variable y saturates and the slope changes. In the case $\sigma < \epsilon$ the influence of the coupling is almost negligible and the second slope is the same we should observe in the 1D case. In other words, as shown in the figure, we have $S^\epsilon(t) \simeq S^\epsilon_y(\infty) + S^\epsilon_x(t)$ where $S^\epsilon_x(t)$ is the entropy calculated from the marginal probability density $\rho_x(x,t) = \int \mathrm{d}y \rho(x,y,t)$. In a similar way $S^\epsilon_y(t)$ is obtained from $\rho_y(y,t) = \int \mathrm{d}x \rho(x,y,t)$.

When $\sigma \approx \epsilon$ the noise on the slowest variable x is enough to spread the distribution as fast as it happens for the variable y so no change in slope is observed.

Thus we observe that, for values of $\sigma > \epsilon$, r_G corresponds quite well to the value of h_{KS} in a broad time interval while for values $\sigma \ll \epsilon$ this correspondence longs only for a very short time. This is noteworthy because this map has pratically the same h_{KS} entropy at the different values of σ used (for $\sigma = 0$ $h_{KS} = -p\log(p) - (1-p)\log(1-p) + \log(r) = 2.22$ while for $\sigma = 0.1$ we numerically obtain $h_{KS} = 2.19$) but with values of $\sigma > \epsilon$ we have $r_G \approx h_{KS}$ while for $\sigma < \epsilon$ the equivalence is lost or longs for a very short time.

It is interesting to compare Fig.3 with Fig.2a: in both cases the "naive" behavior (i.e. $S^\epsilon(t) - S^\epsilon(0) = h_{KS}t$) may have a very short duration and there is a long crossover. The origin of this crossover is due to a sort of "contamination effect" of different times (in other words different mechanisms) involved. For the system (16) this is due to intermittency (the system "feels" λ_+ and λ_-) while for the 2D map it is due to the existence of different times that are relevant at different spatial resolution scales.

In systems with many degrees of freedom the "contamination effect" can produce rather impressive behaviors. As example we can cite the case of fully developed turbulence where, because of the existence of many different characteristic times, the growth of the distance between two trajectories is a power law in time instead to be an exponential (although the system is chaotic) [23].

C. Discretized 1D map

We now discuss a class of dynamical system with zero h_{KS}: the discretized maps also called coarse-grained deterministic automata [24]. We study the discretized (in phase space) version of 1D map defined by the equation

$$n_{t+1} = \eta \left\lfloor \frac{f(n_t)}{\eta} \right\rfloor \tag{23}$$

where η is the discretization parameter (the number of discretized state is $1/\eta$), $\lfloor . \rfloor$ denotes the integer part and $f(x)$ is a tent map:

$$f(x) = \begin{cases} x/p & \text{if } x < p \\ (1-x)/(1-p) & \text{if } p < x < 1. \end{cases} \tag{24}$$

Even if this system is the discretized version of a chaotic map the finiteness of the available states forces the dynamics to become periodic. In ref.[24] it has been observed that the block entropy $H^\epsilon(n)$ defined as in Eq.(2) with $\epsilon \gg \eta$ (i.e. several discrete states in each cell) for small values of n increases with a slope given by the Kolmogorov entropy h_{KS} of the corresponding continuous system $x_{t+1} = f(x_t)$ while for

$$n \gtrsim t_p \simeq -\frac{1}{h_{KS}} \log \eta \tag{25}$$

the block entropy stops increasing thus revealing the periodic nature of the dynamics. We present the behavior of $S^\epsilon(t)$ in this system, varying the value of p in order to observe also the effect of intermittency. As fig.4 shows, the coarse-grained Gibbs'entropy increases as a function of the scaled time $th_{KS}(p)$ with a slope comparable with 1; while it reaches the saturation values at scaled time given approximatively by

$$t_{sat} \simeq \frac{1}{h_{KS}} \log\left(\frac{1}{\epsilon}\right) \approx 7.$$

Note that, in spite of the fact that the KS-entropy of (23) is strictly zero, this behavior is practically the same of the continous system, thus revealing one of the main problem of the use of the coarse-grained Gibbs'entropy

to detect h_{KS}. Indeed the block entropy $H^\epsilon(n)$"feels" that the system is periodic only for $n > t_p$, but

$$h_{KS} t_p \simeq -\log\eta > -\log\epsilon. \tag{26}$$

The last inequality follows from the physical condition we use i.e. that $\epsilon \gg \eta$ in order to have several state inside each cell. The main point is that since $\epsilon \gg \eta$

$$t_p \simeq \frac{1}{h_{KS}} \log(1/\eta)$$

is larger than the saturation time t_{sat}. Therefore *the coarse-grained Gibbs'entropy saturates well before the system feels to be periodic.*

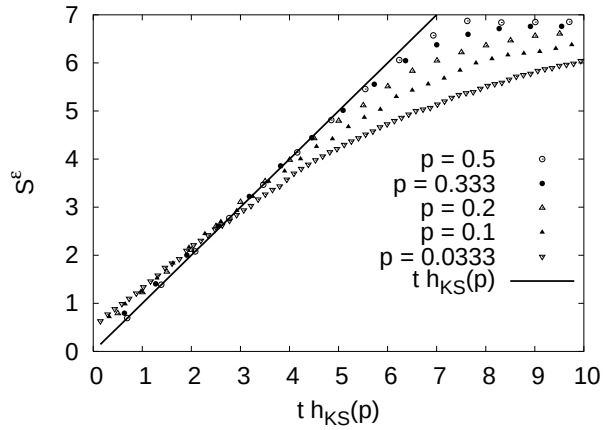

FIG. 4: $S^\epsilon(t)$ as a function of $th_{KS}(p)$ with different values of p, $\epsilon = 10^{-3}$, $1/\eta = 3 \cdot 10^6$

IV. CONCLUSIONS

We have shown that, in spite of some folklore, there is a rather loose relation between the Kolmogorov-Sinai entropy and the growth of the coarse grained (Gibbs like) entropy. Such claimed connection exists only in very special cases, namely systems with a unique characteristic time and very weak intermittency (i.e. small fluctautions of the local Lyapunov exponent). On the contrary, in more interesting (and closer to reality) systems, with multiple characteristic times and/or non negligible intermittency, the relation between h_{KS} and r_G holds (if any) only for a very short time.

The main reason of this is due to the asymptotic nature of h_{KS} (as well as the ϵ-entropy), i.e. its relevance at very large time intervals. On the contrary the growth of the coarse grained entropy only involves short time intervals, and during the early time evolution of $S^\epsilon(t)$ one has entanglement of behaviours at different characteristic space scales.

This phenomenon is rather similar to that observed in the spreading of passive tracers in closed basins [25]. In such a case, if the characteristic length scale of the Eulerian velocities is not very small, compared with the size of the basin, both the diffusion coefficient and the Lyapunov exponent do not give relevant information about the mechanism of spreading.

We stress again that the failure of the relation between h_{KS} and r_G is due to the fact that the growth of the coarse grained entropy is not related to asymptotic properties (i.e. large time and small resolution). This is particularly evident in deterministic discrete states systems that, although non chaotic (i.e. with $h_{KS} = 0$), show a behaviour of $S^\epsilon(t)$ very similar to that observed in genuine chaotic systems (with $h_{KS} \neq 0$).

Acknowledgments

We thank F. Cecconi, P. Grigolini and L. Rondoni for useful discussions.

APPENDIX: A

We discuss here the discontinuity in $S^\epsilon(1)$ for $p \neq 1/2$ observed in fig.1. Indeed for $p = 1/2$ we have $S^\epsilon(1) = \log 2$ while for $p = 1/2 - \xi$, with $\xi \ll 1$, we have $S^\epsilon(1) \simeq 0.85$. Obviously $S^\epsilon(0) = 0$ for all values of p. To understand the reason of this discontinuity let us compute the value of $S^\epsilon(1)$. If we start from the j-th cell we have

$$p^{\epsilon,j}(k,0) = \delta_{k,j}.$$

If $p = 1/2$ at time 1 this cell will be spread on exactly two cells and the trajectory starting at the border of a cell will go exactly on the border of another cell so if, for sake of simplicity, $j < 1/2\epsilon$ (i.e. if the cell is in the region $x < 1/2$), we obtain

$$p^{\epsilon,j}(k,1) = \frac{1}{2}(\delta_{k,2j} + \delta_{k,2j+1})$$

leading to $S^\epsilon(1) = \log 2$ for each j. If we have $p = 1/2 - \xi$ with $\xi \ll 1$ the size of the phase space region populated after one time step is essentially the same. Nevertheless this time the trajectory starting at the border of a cell does not always go to another cell border so in the averaging procedure of Eq.(15) there will be some starting conditions giving approximately a superposition of the cell borders after one time step (for example $i = 0$) and consequently a value of $S^\epsilon(1) \approx \log 2$, but also some starting conditions j resulting in

$$p^{\epsilon,j}(k,1) = \frac{1}{2}\delta_{k,2j+1} + \frac{1}{4}(\delta_{k,2j} + \delta_{k,2j+2}).$$

These terms give a contribution to the average greater than $\log 2$ and lead to the discontinuity of fig.1 when going from $p = 1/2$ to $p \neq 1/2$. Anyway this behavior

is not important for greater values of time so it does not influence the behavior of $S^\epsilon(t)$ for $t > 1$.

In order to check our numerical results we also compute $S^\epsilon(t)$ from $\rho(x,t)$ obtained by the Perron-Frobenius equation. In the case of the tent map we use directly the expression of the formal evolution of $\rho(x,t)$ given by

$$\rho(x,t+1) = p\rho(px,t) + (1-p)\rho(1-x(1-p),t) \quad \text{(A.1)}$$

implemented in a recursive algorithm on a computer program. We fix

$$\rho(x,0) = \begin{cases} 1/\epsilon & \text{if } \mathbf{x} \in \Lambda_k^\epsilon \\ 0 & \text{otherwise} \end{cases}$$

we let $\rho(x,t)$ evolves and then we compute $S^\epsilon(t)$. The results obtained with this method are in perfect agreement with those in Section III and confirm the presence of the discontinuity.

[1] S. Goldstein and J. L. Lebowitz, Physica D **193**, 53 (2004)
[2] C.E. Shannon the Bell Syst. Techn. J. **27**, 379 (1948)
[3] V.I. Arnold and A. Avez *Ergodic Problems of Classical Mechanics* (Benjamin, New York, 1968)
[4] P. Gaspard and X.-J. Wang, Phys. Rep. **235**, 291 (1993)
[5] G. Boffetta, M. Cencini, M. Falcioni and A. Vulpiani, Phys. Rep. **356**, 367 (2002)
[6] H. van Beijeren, Physica D **193**, 90 (2004)
[7] F. Cecconi, D. del Castillo-Negrete, M. Falcioni and A. Vulpiani, Physica D **180**, 129 (2003)
[8] S. Lepri, L. Rondoni and G. Benettin, J. Stat. Phys. **99**, 857 (2001)
[9] C. P. Dettmann and E.G.D. Cohen, J. Stat. Phys. **103**, 589 (2001)
[10] A. K. Pattanayak, P. Brumer, Phys. Rev. Lett. **79**, 4131 (1997)
[11] A. K. Pattanayak, P. Brumer, Phys. Rev. E **56**, 5174 (1997)
[12] A. K. Pattanayak, Phys. Rev. Lett. **83**, 4526 (1999)
[13] P. Gaspard *Chaos, Scattering and Statistical Mechanics* (Cambridge University Press, Cambridge 1998)
[14] J.R. Dorfman *An Introduction to Chaos in Nonequilibrium Statistical Mechanics* (Cambridge university Press, Cambridge 1999)
[15] M. Bologna, P. Grigolini, M. Karagiorgis and A. Rosa, Phys. Rev. E **64**, 016223 (2001)
[16] V. Latora and M. Baranger, Phys. Rev. Lett. **82**, 520 (1999)
[17] J.L. Vega, T. Uzer and J. Ford, Phys. Rev. E **48**, 4314 (1993)
[18] G.M. Zaslavsky and M. A. Edelman, Physica D **193**, 128 (2004)
[19] R.Z. Sagdeev, D.A. Usikov and G.M. Zaslavky *Nonlinear Physics From the Pendulum to Turbulence and Chaos* (Harwood A.P., Chur, 1988)
[20] D. Ruelle *Chaotic Evolution and Strange Attractors* (Cambridge University Press, Cambridge 1989)
[21] P. Manneville, J. Phys. **41**, 1235 (1980)
[22] H. Mori, Byon-Vhyol So and T. Ose, Prog. Theor. Phys. **66**, 1266 (1981)
[23] E. Aurell, G. Boffetta, A. Crisanti, G. Paladin and A. Vulpiani, Phys. Rev. E **53**, 2337 (1996)
[24] M. Falcioni, A. Vulpiani, G. Mantica, S. Pigolotti, Phys. Rev. Lett. **91**, 044101 (2003)
[25] V. Artale, G. Boffetta, A. Celani, M. Cencini and A. Vulpiani, Physics of Fluids A **9**, 3162 (1997)